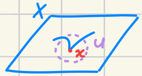
29.06.26

Local Nori fundamental groups and surface singularities



Motivation

Notation $(A, \mathfrak{m})$ normal compl. local $k=\bar{k}$-alg. of dim 2, $k = A/\mathfrak{m}$,
$X = \operatorname{Spec} A$, $U = X \setminus \{x\}$,
where $x := \mathfrak{m} \in X$.

Thm. (Flenner) $\mathbb{Q} \subseteq k$, (X,x) is smooth (A regular) $\Leftrightarrow \pi_1^{ét}(U) = 1$.

Q. What happens in char. p?

Exp. $k = \bar{k}$ of char. 2., $\mu_2 \curvearrowright \hat{\mathbb{A}}^2 = \operatorname{Spec} k[\![x,y]\!]$ diagonally

Then $\hat{\mathbb{A}}^2/\mu_2 =: X$ is singular

in fact $X \simeq \operatorname{Spec} k[\![x,y,z]\!]/(z^2+xy)$.

But for $U = X \setminus \{0\}$ we have

$$\pi_1^{ét}(U) = \pi_1^{ét}(\hat{\mathbb{A}}^2 \setminus \{0\}) \underset{\text{Zar. Nag. purity}}{=} \pi_1^{ét}(\hat{\mathbb{A}}^2) = 1.$$

$\hat{\mathbb{A}}^2$ str. henselian

$\pi^N_{loc}(U,x)$

Problem U has no k-rational points.

Define $\mathcal{C}_{loc}(U,x)$ cat. of triples $(h: V \to U, G, y)$ where

- G is a fin. k-grp. scheme;
- h a G-torsor;
- $y \in Y(k)$ a k-rational point

w.
$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & X. \end{tikzcd}$$
Y: int. closure of $V \to X$.

Prop./Def U admits a local Nori fundamental group scheme: there exist a pro-object "$\varprojlim_{i\in I}$"$(V_i \xrightarrow{h_i} U, G_i, y_i)$ in $\mathcal{C}_{loc}(U,x)$ s.t. for any $(V \to U, G, y)$ in $\mathcal{C}_{loc}(U,x)$ $\exists!$ morphism of pro-objects

$$\text{"}\varprojlim_{i\in I}\text{"}(h_i, G_i, y_i) \longrightarrow (h, G, y). \boxtimes$$

$\to$ same proof as for Nori fundamental group (Ismaele's talk)

Set $\pi^N_{loc}(U,x) = \varprojlim G_i$.

$\pi^N_{loc}(U,X,x)$

generally large!

Have a natural functor

$\rho: \mathcal{C}(X,x) \longrightarrow \mathcal{C}^N_{loc}(U,x)$

pointed torsors $/X$

$$(y \to Y \xrightarrow{G} X) \mapsto (Y \times_X U \to U, G, y)$$

$\Rightarrow \rho_*: \pi^N_{loc}(U,x) \to \pi^N(X,x)$

Def. $\pi^N_{loc}(U,X,x) = \ker \rho_*$

Motto: $\pi^N_{loc}(U,X,x)$ measures $\{$torsors on U not coming from X.

Prop. X regular $\Rightarrow \pi^N_{loc}(U,X,x) = 1$.

Proof chap. II, Prop. 7 of Nori's paper: purity statement for torsors. $\boxtimes$

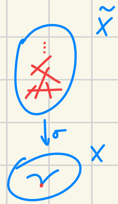
$\to$ every torsor on U extends to one on X

$$\begin{tikzcd} \pi^N_{loc}(U,X,x) \arrow[r, hook] \arrow[dr] & \pi^N_{loc}(U,x) \arrow[r, "\rho_*"] \arrow[d] & \pi^N(X,x) \arrow[d] \\ & \pi_1^{ét}(U) \arrow[r] \arrow[ul, dashed] \arrow[u, dashed] \arrow[ur, "e"] & \pi_1^{ét}(X) \arrow[u, dashed] \end{tikzcd}$$
$\pi_1^{ét}(X) = 1$

Cor. $\pi^N_{loc}(U,X,x)$ finite $\Rightarrow \pi_1^{ét}(U)$ finite group. $\boxtimes$

Thm. $\pi^N_{loc}(U,X,x)$ fin. k-grp. scheme $\Rightarrow (X,x)$ is a rational singular pt.

$\sigma: \tilde{X} \longrightarrow X$ resol'n of X s.t. $\sigma^{-1}(x)_{red} =: C = \bigcup_i C_i$ is s.n.c.



Def. (X,x) is a rational sing. if $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ (i.e $\mathcal{O}_X \xrightarrow{\sim} R\sigma_*\mathcal{O}_{\tilde{X}}$).

Reinterpretation in terms of $\operatorname{Pic}^0(\tilde{X}) = \operatorname{Pic}^0(U)$.

Can construct a specific nilpotent thickening

$$C \hookrightarrow \tilde{X}_{n_0} \hookrightarrow \tilde{X}$$

s.t.
- $H^1(\tilde{X}, \mathcal{O}^\times_{\tilde{X}}) \xrightarrow{\simeq} H^1(\tilde{X}_{n_0}, \mathcal{O}^\times_{\tilde{X}_{n_0}})$
- $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \xrightarrow{\simeq} H^1(\tilde{X}_{n_0}, \mathcal{O}_{\tilde{X}_{n_0}})$.

Define $\mathrm{Pic}(\tilde{X}) := \mathrm{Pic}(\tilde{X}_{n_0}/k)$

loc. fin. type grp. scheme $/k$.

Prop. Have an exact sequence

$$0 \to K \longrightarrow \mathrm{Pic}^0(\tilde{X}) \to \mathrm{Pic}^0(C) \to 0$$

Here

- K smooth conn. unip.;
- $\mathrm{Pic}^0(C)$ s. abelian. var.

In particular, $\mathrm{Pic}^0(\tilde{X})$ is smooth.

Sketch

- $S_0 \subset S$ sq. zero ext'n

$$0 \to \mathcal{I} \to \mathcal{O}_S \to \mathcal{O}_{S_0} \to 0$$

$$\mathcal{I} \cong 1+\mathcal{I}$$

$$\rightsquigarrow 1 \to 1+\mathcal{I} \to \mathcal{O}_S^\times \to \mathcal{O}_{S_0}^\times \to 1$$

$$\Rightarrow \underbrace{H^1(S,\mathcal{I})}_{\simeq \mathbb{G}_a^r} \to \mathrm{Pic}(S) \to \mathrm{Pic}(S_0) \to H^2(S,\mathcal{I})$$

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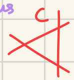
$$0 \to \underbrace{T}_{\text{torus}} \to \mathrm{Pic}^0(C) \to \prod_i \mathrm{Pic}^0(C_i) \to 0$$

e.g.  $\Rightarrow T \simeq \mathbb{G}_m$.

Cor. (X,x) rational $\Leftrightarrow \mathrm{Pic}^0(\tilde{X}) = 0$.

$\to$ have $T_e\mathrm{Pic}^0(\tilde{X}) = H^1(\tilde{X}, \mathcal{O}_{\tilde{X}_{n_0}})$

$= H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

Let $\langle C \rangle \subset \mathrm{Pic}(\tilde{X})$ k-subgroup gen'd by div. w. support on C.

Set

$$\mathrm{Pic}(U) := \mathrm{Pic}(\tilde{X})/\langle C \rangle$$

Fact $\mathrm{Pic}^0(U) = \mathrm{Pic}^0(\tilde{X})$.

Prop. Assume $\exists \ell \neq \mathrm{char}\, k$ a prime s.t. $\pi_1^{\text{ét},ab,\ell}(U)$ finite, then $\mathrm{Pic}^0(U)$ is sm. conn. unipotent.

Proof Str. of $\mathrm{Pic}^0(U)$: $\mathrm{Pic}^0(U)$ sm. conn. unip. $\Leftrightarrow \mathrm{Pic}^0(C) = 0$,

Have a l.e.s

$$0 \to H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \xrightarrow{\simeq} H^1(U, \mathbb{Z}_\ell(1))$$

$$\xrightarrow{\text{purity}} H^2_c(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^2(\tilde{X}, \mathbb{Z}_\ell(1))$$

$$H^2_c(\tilde{X}, \mathbb{Z}_\ell(1)) \cong \bigoplus_i \mathbb{Z}_\ell \cdot [C_i], \qquad H^2(\tilde{X}, \mathbb{Z}_\ell(1)) \underset{\text{Prop. b. change}}{\simeq} H^2(C, \mathbb{Z}_\ell(1)) \cong \bigoplus_i \mathbb{Z}_\ell \cdot [C_i]$$

$$[C_j] \mapsto \sum_i (C_i \cdot C_j)[C_i].$$

$(C_i \cdot C_j)$ is neg. def. $\Rightarrow \det \neq 0$

Then

$$H^1(U, \mathbb{Z}_\ell(1)) \simeq H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \xrightarrow{\simeq} H^1(C, \mathbb{Z}_\ell(1)) = \mathrm{Pic}^0(C)[\ell^\infty].$$

$\mathrm{Pic}^0(C)$ s. abelian

Now $\mathrm{Pic}^0(C) \neq 0 \Rightarrow \mathbb{Z}_\ell \subset \mathrm{Pic}^0(C)[\ell^\infty]$

$\Rightarrow \pi_1^{\text{ét},ab,\ell}(U)$ not fin.

by Hurwitz:

$$\mathbb{Z}_\ell \subset H^1(U, \mathbb{Z}_\ell(1)) = \mathrm{Hom}(\pi_1^{ab,\ell}(U), \mathbb{Z}_\ell(1)). \quad \boxtimes$$

Proof of Thm.

$\pi^N_{loc}(U,X,x)$ fin. grp. scheme

$\Rightarrow \pi_1^{\text{ét},ab,\ell}(U)$ fin.

$\Rightarrow \mathrm{Pic}^0(U)$ sm. conn. unip.

Suppose $\mathrm{Pic}^0(U) \neq 0$ (i.e., (X,x) not rational). Then $\mathrm{Pic}^0(U) \supseteq \mathbb{G}_a$

$\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathrm{Pic}(U))$ is infinite.

Now consider exact sequence

$$0 \to H^1(X, \mu_p) \to H^1(U, \mu_p) \to \mathrm{Hom}(\mathbb{Z}/p, \mathrm{Pic}(U)) \to H^2(X, \mu_p) = 0.$$

$(\mathbb{Z}/p\mathbb{Z})^\vee = \mathrm{Hom}(\mathbb{Z}/p, \mathbb{G}_m) = \mu_p$

X str. henselian

$\Rightarrow$ inf. many μ_p-torsors on U not coming from $X \Rightarrow \pi^N_{loc}(U,X,x)$ not fin. $\boxtimes$

We find $\{\varphi_\lambda : \lambda \in k \setminus \{0\}\} \subseteq \mathrm{Hom}(\mathbb{Z}/p, \mathrm{Pic}(U))$ w. $\varphi_\lambda \neq \varphi_\mu$ for $\lambda \neq \mu$.

Pick for every $\lambda \in k \setminus \{0\}$ a lift $\tilde{\varphi}_\lambda \in H^1(U, \mu_p)$.

Then $\tilde{\varphi}_\lambda$ gives a hom.

$$f_\lambda : \pi^N_{loc}(U,X,x) \subset \pi^N_{loc}(U,x) \to \mu_p$$

and $f_\lambda \neq f_\mu$ for $\lambda \neq \mu$ (otherwise $\varphi_\lambda = \varphi_\mu$). So $\pi^N_{loc}(U,X,x)$ infinite ↯

$\boxtimes$